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# On the number of irreducible components of commuting varieties

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**Abstract**

We will show that for any $n$ and $h,k\in\{0,\dots,n\}$, $h\le k$ the variety of all the pairs $(A,B)$ of $n\times n$ matrices over an algebraically closed field $K$ such that $[A,B]=0$, $\operatorname{rank}A\le k$, $\operatorname{rank}B\le h$ has $\min\{h,n-k\}+1$ irreducible components.Similarly, the corresponding variety of symmetric matrices is reducible if $h,k\in\{1,\dots,n-1\}$ (while it is irreducible if $h$ is 0 and if $\operatorname{char}K\ne 2$ and $k$ is $n$); if $\operatorname{char}K\ne 2$ and $h,k$ are even the corresponding variety of antisymmetric matrices is reducible if $h,k\in\{2,\dots,n-1\}$ (while it is irreducible if $h$ is 0 and if $\operatorname{char}K=0$ and $k$ is $n$ or $n-1$). © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

We denote by $M(n,K)$ the Lie algebra of all $n\times n$ matrices over an algebraically closed field $K$, by $S(n,K)$ and $A(n,K)$, respectively, its subspace of all the symmetric matrices and its Lie subalgebra of all the antisymmetric matrices. We regard these spaces as affine spaces and, for $h,k=0,\dots,n$, we set:

$$\mathscr{M}(n,K)_{k,h}=\{(A,B): A,B\in M(n,K),[A,B]=0,\operatorname{rank}A\le k,\ \operatorname{rank}B\le h\},$$

$$\mathscr{S}(n,K)_{k,h}=\mathscr{M}(n,K)_{k,h}\cap(S(n,K)\times S(n,K)),$$

$$\mathscr{A}(n,K)_{k,h}=\mathscr{M}(n,K)_{k,h}\cap(A(n,K)\times A(n,K)).$$

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We may assume $h \leq k$. In [1] we proved the following theorems.

**Theorem 1.1.** *$\mathcal{M}(n,K)_{n,h}$ is irreducible of dimension $n^2+h$ for all $n$ and $h=0,\dots,n$. $\mathcal{S}(n,K)_{n,h}$ is irreducible of dimension $n(n+1)/2+h$ for all $n$ and $h=0,\dots,n$ if* $\operatorname{char} K \neq 2$, *for all $n$ and $h=0$ if* $\operatorname{char} K = 2$.

**Theorem 1.2.** *$\mathcal{A}(n,K)_{n,h}$ is irreducible for all $n$ and $h=0,\dots,n$ if* $\operatorname{char} K = 0$, *for all $n$ and $h=0,1$ if* $\operatorname{char} K \neq 2$. *If $h$ is even it has dimension $n(n-1)/2+h/2$.*

The irreducibility of $\mathcal{M}(n,K)_{n,n}$ was before proved in [6] and in [11]. In [14] this result was extended to any reductive Lie algebra over a field K of characteristic O, so including the irreducibility of $\mathcal{A}(n,K)_{n,n}$. The irreducibility and other properties of $\mathcal{S}(n,k)_{n,n}$ and of some generalizations of it were proved in [2]. In [13] many properties of more general "commuting varieties" were proved, including the irreducibility and the normality of $\mathcal{S}(n,K)_{n,n}$, which was proved also in [3] of some varieties having $\mathcal{M}(n,K)_{n,n}$ as a particular case was studied in [8], [9], [12] and in [7].

We denote by $M(n,K)_h$ the variety of all $A \in M(n,K)$ such that $\operatorname{rank} A \leq h$ and set $S(n,K)_h = S(n,K) \cap M(n,K)_h$, $A(n,K)_h = A(n,K) \cap M(n,K)_h$. The following result is due to Kempf [10], see also [1, Theorem 2.1] and to Weyl [16], De Concini and Procesi [4].

**Theorem 1.3** (Kempf [10]). *$M(n,K)_h$, $S(n,K)_h$ and $A(n,K)_h$ are irreducible varieties of dimension $2nh-h^2$, $nh-h(h-1)/2$ and, if $h$ is even, $nh-h(h+1)/2$, respectively.*

If $h=0$ (or $h=1$ if $\operatorname{char} K \neq 2$ for antisymmetric matrices) the varieties $\mathcal{M}(n,K)_{k,h}$, $\mathcal{S}(n,K)_{k,h}$ and $\mathcal{A}(n,K)_{k,h}$ are irreducible by Theorem 1.3.

For any $A \in M(n,K)$ let $p_A(x) \in K[x]$ be the minimum polynomial of $A$ and let $L_A$ be the corresponding endomorphism of $K^n$. For $k=0,\dots,n$ let

$$Z(n,K)_k = \{A \in M(n,K)_k\colon\ \operatorname{rank} A = k,\ p_A(x) = f(x) \text{ or}$$
$$p_A(x) = x f(x),\ \text{where } f(x) \in K[x],\ f(0) \neq 0\}.$$

By the elimination theorem $Z(n,K)_k$ is an open subset of $M(n,K)_k$. We set

$$\mathcal{Z}(n,K)_{k,h} = \{(A,B) \in \mathcal{M}(n,K)_{k,h}\colon A \in Z(n,K)_k\}$$

and for $t=0,\dots,\min\{h,n-k\}$ we set

$$\mathcal{Z}^t(n,K)_{k,h} = \{(A,B) \in \mathcal{Z}(n,K)_{k,h}\colon\ \dim(\ker L_A \cap \ker L_B) \geq n-k-t,$$
$$\operatorname{rank}(AB) \leq h-t\}.$$

We will prove the following results.

**Theorem 1.4.** *For all $n$ and $h,k \in \{0,\dots,n\}$, $h \le k$ the irreducible components of $\mathscr{M}(n,K)_{k,h}$ are $\overline{\mathscr{L}^t(n,k)_{k,h}}$, $t=0,\dots,\min\{h,n-k\}$. They have dimension $2nk-k^2+2(n-k)t-t^2+h-t$.*

**Theorem 1.5.** *For all $n$ and $h,k \in \{1,\dots,n-1\}$ $\mathscr{S}(n,K)_{k,h}$ is reducible; if* $\operatorname{char} K \neq 2$ *$\mathscr{A}(n,K)_{k,h}$ is reducible for all $h,k \in \{2,\dots,n-1\}$ if $n$ is even, for all $h,k \in \{2,\dots,n-2\}$ if $n$ is odd.*

## 2. The reducibility of $\mathscr{M}(n,K)_{k,h}$, $\mathscr{S}(n,K)_{k,h}$ and $\mathscr{A}(n,K)_{k,h}$

Theorems 1.4 and 1.5 will be a consequence of the following two propositions.

**Proposition 2.1.** *The irreducible components of $\mathscr{L}(n,K)_{k,h}$ are $\mathscr{L}^t(n,K)_{k,h}$, $t=0,\dots,\min\{h,n-k\}$; they have dimension $2nk-k^2+2(n-k)t-t^2+h-t$. Moreover, we have*

$$\mathscr{S}(n,K)_{k,h} \cap \mathscr{L}(n,K)_{k,h} = \bigcup_{t \in \{0,\dots,\min\{h,n-k\}\}} (\mathscr{S}(n,K)_{k,h} \cap \mathscr{L}^t(n,K)_{k,h}),$$

*where each term of this union is not contained in the union of the others and if* $\operatorname{char} K \neq 2$ *it is irreducible. If* $\operatorname{char} K \neq 2$ *and $h,k,t,t'$ are even we have*

$$\mathscr{A}(n,K)_{k,h} \cap \mathscr{L}(n,K)_{k,h} = \bigcup_{t \in \{0,\dots,\min\{h,n-k\}\}} (\mathscr{A}(n,K)_{k,h} \cap \mathscr{L}^t(n,K)_{k,h}),$$

*where each term of this union is not contained in the union of the others and if* $\operatorname{char} K = 0$ *it is irreducible.*

**Proposition 2.2.** *For any $h,k \in \{0,\dots,n\}$ $\mathscr{L}(n,K)_{k,h}$ is a dense subset of $\mathscr{M}(n,K)_{k,h}$.*

In the proof of Propositions 2.1 and 2.2 we will use the following two lemmas. They were proved by Turnbull and Aitken [15, Chapter X] and by Gantmacher [5, Chapter VIII, Section 2].

**Lemma 2.3** (Turnbull and Aitken [15]). *Let $A \in M(n,K)$ be a block diagonal matrix whose diagonal blocks are $A_1,\dots,A_k$ where $A_i \in M(n_i,K)$ for $i=1,\dots,k$ and $p_{A_1}(x),\dots,p_{A_k}(x)$ are relatively prime. If $B \in Z_A$ then $B$ is a block diagonal matrix whose diagonal blocks $B_1,\dots,B_k$ are such that $B_i \in M(n_i,K)$ for $i=1,\dots,n$.*

**Lemma 2.4** (Turnbull and Aitken [15]). *Let $N \in M(n,K)$ be nilpotent; let $u_1 \ge \cdots \ge u_t$ be the orders of the Jordan blocks of $N$ and $\Delta = \{v_j^{u_j-1},\dots,v_j^0,\ j=1,\dots,t\}$ be a basis of $K^n$ such that the matrix of $L_N$ with respect to $\Delta$ is in Jordan canonical form (then $v_j^r = L_N^r v_j^0$ for $r=1,\dots,u_j-1$). Let us regard the matrix of $L_A$ with respect to $\Delta$ as* *a block matrix* $(A_{jk})_{jk}$, $j,k=1,\ldots,t$, *where* $A_{jk}$ *is an* $u_j \times u_k$ *matrix. Then* $A \in Z_N$ *if and only if for any* $j,k \in \{1,\ldots,t\}$, $j \geq k$ $A_{jk}$ *and* $A_{kj}$ *have the following form*:

$$A_{jk} = \begin{pmatrix} 0 & \ldots & 0 & a_{jk}^1 & a_{jk}^2 & \ldots & a_{jk}^{u_j} \\ 0 & \ldots & \ldots & 0 & a_{jk}^1 & \ldots & a_{jk}^{u_j-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & a_{jk}^1 \end{pmatrix},$$

$$A_{kj} = \begin{pmatrix} a_{kj}^1 & a_{kj}^2 & \ldots & a_{kj}^{u_j} \\ 0 & a_{kj}^1 & \ldots & a_{kj}^{u_j-1} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & a_{kj}^1 \\ 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 \end{pmatrix},$$

*where for* $u_j = u_k$ *we omit the first* $u_k - u_j$ *columns and the last* $u_k - u_j$ *rows, respectively*.

**Proof of Proposition 2.1.** Let

$$\mathscr{V}(n,K)_{k,h} = \left\{(A,B) \in \mathscr{M}(n,K)_{k,h}\colon\ A = \begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix},\right.$$

$$\left. B = \begin{pmatrix} B'' & 0 \\ 0 & B' \end{pmatrix}, A', B' \in M(k,K), \det A' \neq 0 \right\}.$$

Then by Lemma 2.3 $(A,B) \in \mathscr{Z}(n,K)_{k,h}$ if and only if there exists $G \in \mathrm{GL}\,(n,K)$ such that $(G^{-1}AG,\ G^{-1}BG) \in \mathscr{V}(n,K)_{k,h}$. By [5, Chapter XI, Section 3, Corollary 2, Section 4, Corollary] if $(A,B) \in \mathscr{S}(n,K)_{k,h}$ or $\mathrm{char}\, K \neq 2$, $h,k$ are even and $(A,B) \in \mathscr{A}(n,K)_{k,h}$ then $(A,B) \in \mathscr{Z}(n,K)_{k,h}$ if and only if there exists $O \in \mathrm{SO}\,(n,K)$ such that $(O^tAO, O^tBO) \in \mathscr{V}(n,K)_{k,h}$.

Since $\mathscr{V}(n,K)_{k,h}$ is the union of the following subvarieties:

$$\begin{aligned} \mathscr{V}^t(n,K)_{k,h} &= \{(A,B) \in \mathscr{V}(n,K)_{k,h}\colon \mathrm{rank}\, B'' \leq t,\ \mathrm{rank}\, B' \leq h-t\} \\ &= \{(A,B) \in \mathscr{V}(n,K)_{k,h}\colon \dim(\ker L_A \cap \ker L_B) \geq n-k-t, \\ &\qquad \mathrm{rank}\,(AB) \leq h-t\}, \quad t = 0,\ldots,\min\{h, n-k\}, \end{aligned}$$

$\mathscr{Z}(n,K)_{k,h}$ is the union of the subvarieties $\mathscr{Z}^t(n,K)_{k,h}$, $t=0,\ldots,\min\{h,n-k\}$. But for any $t \in \{0,\ldots,\min\{h,n-k\}\}$ the set

$$\{(A,B) \in \mathscr{Z}(n,K)_{k,h}\colon\ \dim(\ker L_A \cap \ker L_B) \leq n-k-t, \mathrm{rank}\,(AB) \geq h-t\}$$

is a non-empty open subset of $\mathscr{Z}(n,K)_{k,h}$ which has empty intersection with

$$\bigcup_{t' \in (\{0,\dots,\min\{h,n-k\}\}\setminus\{t\})} \mathscr{Z}^{t'}(n,K)_{k,h}.$$

By Theorems 1.1 and 1.3 for $t = 0,\dots,\min\{h, n-k\}$ the variety $\mathscr{V}^t(n,K)_{k,h}$ is irreducible. Since $\mathscr{Z}^t(n,K)_{k,h}$ is the image of a morphism from $\mathrm{GL}(n,K) \times \mathscr{V}^t(n,K)_{k,h}$, it is irreducible.

Let us consider the projection of $\mathscr{Z}^t(n,K)_{k,h}$ on $Z(n,K)_k$. If $A$ belongs to the open subset of $Z(n,K)_k$ of all the matrices which have $k+1$ different eigenvalues then the irreducible components of the fiber of $A$ have dimension $2(n-k)t - t^2 + h - t$. Hence $\mathscr{Z}^t(n,K)_{k,h}$ has dimension $2nk - k^2 + 2(n-k)t - t^2 + h - t$.

We get the claim for symmetric and antisymmetric matrices in the same way, by using also Theorem 1.2. □

**Proof of Proposition 2.2.** We proceed by induction on $n$, since the claim is obvious for $n = 1$. Let $n > 1$. By Theorems 1.1 and 1.3 the claim is true if $h$ or $k$ is 0 or $n$, hence we assume $h, k \neq 0, n$.

Let $\mathscr{A}$ be an open subset of $\mathscr{M}(n,K)_{k,h}$ and $(A,B) \in \mathscr{A}$; we want to prove that there exists $(\hat{A},\hat{B}) \in \mathscr{A}$ such that $\hat{A}$ or $\hat{B}$ is not nilpotent. In fact, then by Lemma 2.3 we may apply the inductive hypothesis and get the claim. Let $A$ and $B$ be nilpotent. If $B=0$, since $M(n,K)_k \times \{0\} \subset \mathscr{M}(n,K)_{k,h}$ we get the claim by Theorem 1.3. Let $B \neq 0$ and let $u_1 \geq \cdots \geq u_t$ be the orders of the Jordan blocks of $B$. We may assume that $B$ is in Jordan canonical form and regard $A$ to be a block matrix $(A_{jl})_{jl}$, $j, l = 1,\dots,t$, as in Lemma 2.4. Let $x \in K$. Suppose that there exist $j, l \in \{1,\dots,t\}$ such that $u_j = u_l = u_1$ and $\operatorname{rank} A_{jl} = u_1$. For any $s \in \{1,\dots,t\}$ let $A_s$ and $A^s$ be, respectively, the $s$th row of blocks of $A$ and the $s$th column of blocks of $A$. If $j \leq l$ and we add $xA^l$ to $A^j$ we get a matrix which has the same rank as $A$ and, by Lemma 2.4, commutes with $B$, but for a general choice of $x$ it is not nilpotent (since the diagonal entries of its block of indices $(j,j)$ depend on $x$). The same if $j > l$ and we add $xA_j$ to $A_l$. Suppose that for any $j, l \in \{1,\dots,t\}$ such that $u_j = u_l = u_1$ $\operatorname{rank} A_{jl} < u_1$. If we substitute the entry of indices $(1,1)$ of $B$ with $x$ and we add to any row of $A$ of index belonging to $\{2,\dots,u_1\}$ the previous row multiplied by $-x$, we get two matrices which have, respectively, the same ranks as $B$ and $A$ and commute, but for $x \neq 0$ the first one is not nilpotent. By these observations we get the claim. □

As a corollary of the proof of Proposition 2.2 we get the following result.

**Corollary 2.5.** *The subset of $\mathscr{M}(n,K)_{k,h}$ of all the pairs of simultaneously diagonalizable matrices is dense.*

**Proof.** By Theorems 1.1 and 1.3 if $h$ or $k$ is 0 or $n$ the claim is true. Otherwise, since by the proof of Proposition 2.2 the subset of $\mathscr{M}(n,K)_{k,h}$ of all the pairs $(A,B)$ such that $A$ or $B$ is not nilpotent is dense, we get the claim by Lemma 2.3 and induction on $n$. □

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